A characterization theorem for the L^2-discrepancy of integer points in dilated polygons

G. Travaglini M. R. Tupputi

Abstract

Let C be a convex d-dimensional body. If ρ is a large positive number, then the dilated body ρC contains $\rho^d |C| + \mathcal{O}\left(\rho^{d-1}\right)$ integer points, where $|C|$ denotes the volume of C. The above error estimate $\mathcal{O}\left(\rho^{d-1}\right)$ can be improved in several cases. We are interested in the L^2-discrepancy $D_C(\rho)$ of a copy of ρC thrown at random in $\mathbb{R}^d$. More precisely, we consider

$$D_C(\rho) := \left\{ \int_{\mathbb{T}^d} \int_{SO(d)} \left| \mathrm{card}\left(\left(\rho\sigma(C) + t\right) \cap \mathbb{Z}^d \right) - \rho^d |C| \right|^2 d\sigma dt \right\}^{1/2} ,$$

where $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$ is the d-dimensional flat torus and $SO(d)$ is the special orthogonal group of real orthogonal matrices of determinant 1.

An argument of D. Kendall shows that $D_C(\rho) \leq c\ \rho^{(d-1)/2}$. If C also satisfies the reverse inequality $D_C(\rho) \geq c_1\ \rho^{(d-1)/2}$, we say that C is *L^2-regular*. L. Parnovski and A. Sobolev proved that, if $d > 1$, a d-dimensional unit ball is L^2-regular if and only if $d \not\equiv 1 \pmod 4$.

In this paper we characterize the L^2-regular convex polygons. More precisely we prove that a convex polygon is not L^2-regular if and only if it can be inscribed in a circle and it is symmetric about the centre.

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1 Introduction

We identify the d-dimensional flat torus $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$ with the unit cube $\left[-\frac{1}{2}, \frac{1}{2}\right)^d$ and we recall that a sequence $\{t_j\}_{j=1}^{+\infty} \subset \mathbb{T}^d$ is *uniformly distributed* if one of the following three equivalent conditions is satisfied: (i) for every d-dimensional box $I \subset \left[-\frac{1}{2}, \frac{1}{2}\right)^d$ with volume $|I|$,

$$\lim_{N \to +\infty} \frac{1}{N} \mathrm{card}\{t_j \in I : 1 \leq j \leq N\} = |I| \ ;$$

(ii) for every continuous function f on $\mathbb{T}^d$,

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{j=1}^{N} f(t_j) = \int_{\mathbb{T}^d} f(t)\ dt \ ;$$

and (iii) for every $0 \neq k \in \mathbb{Z}^d$,

$$\lim_{N\to+\infty} \frac{1}{N} \sum_{j=1}^{N} e^{2\pi i k \cdot t_j} = 0 ,$$

where "$\cdot$" denotes the d-dimensional inner product.

The concept of uniform distribution and the defining properties given above go back to a fundamental paper written one hundred years ago by H. Weyl [34]; see [25] for the basic reference on uniformly distributed sequences. Observe that the above definition does not show the quality of a uniformly distributed sequence. In the late thirties J. van der Corput coined the term *discrepancy*: let $\mathfrak{D}_N := \{t_j\}_{j=1}^N$ be a sequence of N points in $\mathbb{T}^d$, henceforth called a *distribution* (of N points), and let

$$D(\mathfrak{D}_N) := \sup_{I\subset\mathbb{T}^d} \left|\mathrm{card}\left(\{t_j\}_{j=1}^N \cap I\right) - N|I|\right|$$

be the (non normalized) discrepancy associated with $\mathfrak{D}_N$ with respect to the d-dimensional boxes I in $\mathbb{T}^d$. There are different approaches to define a discrepancy that measures the quality of a distribution of points; see e.g. [2, 12, 19, 25, 26, 18] for an introduction of discrepancy theory. See [4, 14, 15, 16] for the connections of discrepancy to energy and numerical integration.

Throughout this paper we shall denote by $c, c_1, \ldots$ positive constants which may change from step to step.

K. Roth [31] proved the following lower estimate: for every distribution $\mathfrak{D}_N$ of N points in $\mathbb{T}^2$, we have

$$\int_{\mathbb{T}^2} \left|\mathrm{card}(\mathfrak{D}_N \cap I_{x,y}) - Nxy\right|^2 \, dxdy \geq c \, \log N , \tag{1}$$

where $I_{x,y} := [0,x] \times [0,y]$ and $0 \leq x, y < 1$. This yields $D(\mathfrak{D}_N) \geq c \, \log^{1/2} N$. H. Davenport [17] proved that the estimate (1) is sharp.

W. Schmidt [32] investigated the discrepancy with respect to discs. His results were improved and extended, independently, by J. Beck [1] and H. Montgomery [27]: for every convex body $C \subset \left[-\frac{1}{2}, \frac{1}{2}\right)^d$ of diameter less than one and for every distribution $\mathfrak{D}_N$ of N points in $\mathbb{T}^d$, one has

$$\int_0^1 \int_{SO(d)} \int_{\mathbb{T}^d} \left|\mathrm{card}(\mathfrak{D}_N \cap (\lambda\sigma(C) + t)) - \lambda^d N|C|\right|^2 \, dtd\sigma d\lambda \geq c \, N^{(d-1)/d} . \tag{2}$$

This relation implies that for every distribution $\mathfrak{D}_N$ there exists a translated, rotated, and dilated copy $\overline{C}$ of a given convex body $C \subset \left[-\frac{1}{2}, \frac{1}{2}\right)^d$ having diameter less than one, such that

$$\left|\mathrm{card}(\mathfrak{D}_N \cap \overline{C}) - N|\overline{C}|\right| \geq c \, N^{(d-1)/(2d)} .$$

J. Beck and W. Chen [3] proved that (2) is sharp. Indeed, they showed that for every positive integer N there exists a distribution $\widetilde{\mathfrak{D}}_N \subset \mathbb{T}^d$ satisfying

$$\int_{SO(d)} \int_{\mathbb{T}^d} \left|\mathrm{card}(\widetilde{\mathfrak{D}}_N \cap C) - N|C|\right|^2 \, dtd\sigma \leq c \, N^{(d-1)/d} . \tag{3}$$

This distribution $\widetilde{\mathfrak{D}}_N$ can be obtained either by applying a probabilistic argument or by reduction to a lattice point problem; see [7, 11, 13, 33] for a comparison of probabilistic and deterministic results.

In the following, we shall consider bounds for the integral in (3) for distributions of N points that are restrictions of a shrunk integer lattice to the unit cube $\left[-\frac{1}{2},\frac{1}{2}\right)^d$. Due to an argument in [9, p. 3533] that also extends to higher dimensions, we may assume that N is a dth power $N = M^d$ for a positive integer M. More precisely, we consider distributions

$$\mathfrak{D}_N := \left(\frac{1}{N^{1/d}}\mathbb{Z}^d\right) \cap \left[-\frac{1}{2},\frac{1}{2}\right)^d \ .$$

Given a convex body $C \subset \left[-\frac{1}{2},\frac{1}{2}\right)^d$ of diameter less than one, we then have

$$\operatorname{card}\left(\mathfrak{D}_N \cap C\right) - N\left|C\right| = \operatorname{card}\left(\mathbb{Z}^d \cap N^{1/d}C\right) - N\left|C\right| \ . \tag{4}$$

Estimation of the RHS in (4) is a classical lattice point problem. Results concerning lattice points are extensively used in different areas of pure and applied mathematics; see, for example, [20, 21, 24].

For the definition of a suitable discrepancy function, we change the discrete dilation $N^{1/d}$ in (4) to an arbitrary dilation $\rho \geq 1$ and replace the convex body C in (4) with a translated, rotated and then dilated copy $\rho\sigma\left(C\right) + t$, where $\sigma \in SO\left(d\right)$ and $t \in \mathbb{T}^d$. Thus the discrepancy

$$D_C^\rho(\sigma,t) := \operatorname{card}(\mathbb{Z}^d \cap (\rho\sigma(C)+t)) - \rho^d|C| = \sum_{k\in\mathbb{Z}^d} \chi_{\rho\sigma(C)+t}(k) - \rho^d|C|$$

is defined as the difference between the number of integer lattice points in the set $\rho\sigma\left(C\right)+t$ and its volume $\rho^d\left|C\right|$ (here, χ_A denotes the characteristic function for the set A). It is easy to see (e.g., [7]) that the periodic function $t \mapsto D_C^\rho(\sigma,t)$ has the Fourier series expansion

$$\rho^d \sum_{0\neq m\in\mathbb{Z}^d} \widehat{\chi_{\sigma(C)}}\left(\rho m\right)\ e^{2\pi i m\cdot t} \ . \tag{5}$$

D. Kendall [22] seems to have been the first to realize that multiple Fourier series expansions can be helpful in certain lattice point problems. Using our notation, he proved that for every convex body $C \subset \mathbb{R}^d$ and $\rho \geq 1$

$$\|D_C^\rho\|_{L^2(SO(d)\times\mathbb{T}^d)} \leq c\ \rho^{(d-1)/2} \ . \tag{6}$$

This also follows from more recent results in [30] and [8] as demonstrated next. Given a convex body $C \subset \mathbb{R}^d$, we define the (spherical) average decay of $\widehat{\chi_C}$ as

$$\|\widehat{\chi_C}\left(\rho\cdot\right)\|_{L^2(\Sigma_{d-1})} := \left\{\int_{\Sigma_{d-1}} \left|\widehat{\chi_C}\left(\rho\tau\right)\right|^2\ d\tau\right\}^{1/2} \ ,$$

where $\Sigma_{d-1} := \left\{t \in \mathbb{R}^d : |t| = 1\right\}$ and τ is the rotation invariant normalized measure on Σ_{d-1}. Extending an earlier result of A. Podkorytov [30], L. Brandolini, S. Hofmann, and A. Iosevich [8] proved that

$$\|\widehat{\chi_C}\left(\rho\cdot\right)\|_{L^2(\Sigma_{d-1})} \leq c\ \rho^{-(d+1)/2} \ . \tag{7}$$

By applying the Parseval identity to the Fourier series (5) of the discrepancy function, we obtain Kendall's result (6); i.e.,

$$\begin{aligned} \|D_C^\rho\|^2_{L^2(SO(d)\times\mathbb{T}^d)} &= \rho^{2d} \sum_{0\neq k\in\mathbb{Z}^d} \int_{SO(d)} \left|\widehat{\chi_{\sigma(C)}}\,(\rho k)\right|^2 \, d\sigma \\ &\leq c\ \rho^{2d} \sum_{0\neq k\in\mathbb{Z}^d} |\rho k|^{-(d+1)} \leq c_1\ \rho^{d-1}\ . \end{aligned} \tag{8}$$

We are interested in the reversed inequality

$$\|D_C^\rho\|^2_{L^2(SO(d)\times\mathbb{T}^d)} \geq c_1\ \rho^{d-1}\ , \tag{9}$$

which, as we shall see, may or may not hold. To understand this, let us assume that (7) can be reversed

$$\|\widehat{\chi_C}\,(\rho\cdot)\|_{L^2(\Sigma_{d-1})} \geq c_1\ \rho^{-(d+1)/2}\ . \tag{10}$$

This relation (10) is true for a simplex (see [6, Theorem 2.3]) but it is not true for every convex body (see the next section).

The following result was proved in [6, Proof of Theorem 3.7].

Proposition 1 *Let C in $\mathbb{R}^d$ be a convex body which satisfies (10). Then C satisfies (9).*

Proof. Indeed,

$$\begin{aligned} \|D_C^\rho\|^2_{L^2(SO(d)\times\mathbb{T}^d)} &= \rho^{2d} \sum_{0\neq k\in\mathbb{Z}^d} \int_{SO(d)} \left|\widehat{\chi_{\sigma(C)}}\,(\rho k)\right|^2 \, d\sigma \\ &\geq c\ \rho^{2d} \int_{SO(d)} \left|\widehat{\chi_{\sigma(C)}}\,(\rho k')\right|^2 \, d\sigma \geq c_1\ \rho^{d-1}\ , \end{aligned} \tag{11}$$

where k' is any non-zero element in $\mathbb{Z}^d$. ■

We are going to see that (9) does not imply (10).

2 L^2-regularity of convex bodies

We say that a convex body $C \subset \mathbb{R}^d$ is L^2-*regular* if there exists a positive constant c_1 such that

$$c_1\ \rho^{(d-1)/2} \leq \|D_C^\rho\|_{L^2(SO(d)\times\mathbb{T}^d)} \tag{12}$$

(by (6) we already know that $\|D_C^\rho\|_{L^2(SO(d)\times\mathbb{T}^d)} \leq c_2\ \rho^{(d-1)/2}$ for some $c_2 > 0$). If (12) fails we say that C is L^2-*irregular*.

Let $d > 1$. L. Parnovski and A. Sobolev [29] proved that the d-dimensional ball $B_d := \left\{t \in \mathbb{R}^d : |t| \leq 1\right\}$ is L^2-regular if and only if $d \not\equiv 1 \pmod 4$.

More generally, it was proved [5] that if $C \subset \mathbb{R}^d$ $(d > 1)$ is a convex body with smooth boundary, having everywhere positive Gaussian curvature, then (i) if C is not symmetric about a point, or if $d \not\equiv 1 \pmod 4$, then C is L^2-regular; (ii) if C is symmetric about a point and if $d \equiv 1 \pmod 4$ then C is L^2-irregular.

L. Parnovski and N. Sidorova [28] studied the above problem for the non-convex case of a d-dimensional annulus ($d > 1$). They provided a complete answer in terms of the width of the annulus.

In the case of a polyhedron P, inequality (6) was extended to L^p norms in [6]: for any $p > 1$ and $\rho \geq 1$ we have

$$\|D_P^\rho\|_{L^p(SO(d)\times\mathbb{T}^d)} \leq c_p \ \rho^{(d-1)(1-1/p)}$$

and, specifically for simplices S, one has

$$c_p' \ \rho^{(d-1)(1-1/p)} \leq \|D_S^\rho\|_{L^p(SO(d)\times\mathbb{T}^d)} \leq c_p \ \rho^{(d-1)(1-1/p)} \ .$$

In particular, this implies that the d-dimensional simplices are L^2-regular.

For the planar case it was proved in [10, Theorem 6.2] that every convex body with piecewise C^∞ boundary that is not a polygon is L^2-regular.

Related results can be found in [6, 13, 23].

Until now no example of a L^2-irregular polyhedron has been found.

We are interested in identifying the L^2-regular convex polyhedrons. In this paper we give a complete answer for the planar case.

Let us first compare the L^2-regularity for a disc $B \subset \mathbb{R}^2$ and a square $Q \subset \mathbb{R}^2$. Their characteristic functions χ_B and χ_Q do not satisfy (10). Indeed, $\widehat{\chi_B}(\xi) = |\xi|^{-1} J_1(2\pi|\xi|)$, where J_1 is the Bessel function (see e.g. [33]). Then the zeroes of J_1 yield an increasing diverging sequence $\{\rho_u\}_{u=1}^\infty$ such that

$$\|\widehat{\chi_C}(\rho_u\cdot)\|_{L^2(\Sigma_1)} = 0 \ .$$

Less obvious is the fact that the inequality $\|\widehat{\chi_C}(\rho\cdot)\|_{L^2(\Sigma_1)} \geq c\ \rho^{-3/2}$ fails for a square Q: it was observed in [6] the existence of a positive constant c such that, for every positive integer n, one has

$$\|\widehat{\chi_Q}(n\cdot)\|_{L^2(\Sigma_1)} \leq c\ n^{-7/4} \ . \tag{13}$$

For completeness we write the short proof of (13). Indeed, let $Q = \left[-\frac{1}{2}, \frac{1}{2}\right]^2$ and let n be a positive integer. Let $\Theta := (\cos\theta, \sin\theta)$. Then an explicit computation of $\widehat{\chi_Q}$ yields

$$\begin{aligned}
&\int_0^{2\pi} |\widehat{\chi_Q}(n\Theta)|^2 \ d\theta = 8\int_0^{\pi/4} \left|\frac{\sin(\pi n\cos\theta)}{\pi n\cos\theta}\frac{\sin(\pi n\sin\theta)}{\pi n\sin\theta}\right|^2 \ d\theta \\
&\leq c\,\frac{1}{n^4}\int_0^{\pi/4}\left|\frac{\sin(\pi n\cos\theta)}{\sin\theta}\right|^2 \ d\theta = c\,\frac{1}{n^4}\int_0^{\pi/4}\left|\frac{\sin\left(\pi n\left(1-2\sin^2(\theta/2)\right)\right)}{\sin\theta}\right|^2 \ d\theta \\
&\leq c'\,\frac{1}{n^4}\int_0^{\pi/4}\left|\sin(2\pi n\sin^2(\theta/2))\right|^2\theta^{-2} \ d\theta \\
&\leq c''\,\frac{1}{n^4}\int_0^{n^{-1/2}} n^2\theta^2 \ d\theta + c''\ \frac{1}{n^4}\int_{n^{-1/2}}^{\pi/4}\theta^{-2} \ d\theta \leq c'''\ n^{-7/2} \ .
\end{aligned}$$

Then B and Q *may* be L^2-irregular.

On the one hand it is known that a disc B is L^2-regular (see [29] or [10, Theorem 6.2]), so that (9) does not imply (10). On the other hand we shall prove in this paper that Q is L^2-irregular.

The L^2-irregularity of the square Q is shared by each member of the family of polygons described in the following definition.

Definition 2 *Let $\mathfrak{P}$ be the family of all convex polygons in $\mathbb{R}^2$ which can be inscribed in a circle and are symmetric about the centre.*

3 Statements of the results

We now state our main result.

Theorem 3 *A convex polygon P is L^2-regular if and only if $P \notin \mathfrak{P}$.*

The "only if" part is a consequence of the following more precise result.

Proposition 4 *If $P \in \mathfrak{P}$, then for every $\varepsilon > 0$ there is an increasing diverging sequence $\{\rho_u\}_{u=1}^{\infty}$ such that*

$$\|D_P^{\rho_u}\|_{L^2(SO(2)\times\mathbb{T}^2)} \leq c_\varepsilon \ \rho_u^{1/2} \log^{-1/(32+\varepsilon)}(\rho_u) \ .$$

Theorem 3 above and [10, Theorem 6.2] yield the following more general result.

Corollary 5 *Let C be a convex body in $\mathbb{R}^2$ having piecewise smooth boundary. Then C is not L^2-regular if and only if it belongs to $\mathfrak{P}$.*

The following result shows that Theorem 3 is essentially sharp.

Proposition 6 *For every $P \in \mathfrak{P}$, for $\varepsilon > 0$ arbitrary small, and for any ρ large enough,*

$$\|D_P^{\rho}\|_{L^2(SO(2)\times\mathbb{T}^2)} \geq c_\varepsilon \ \rho^{1/2-\varepsilon} \ ,$$

where c_ε is independent of ρ.

The "if" part of Theorem 3 is a consequences of the following three lemmas.

Lemma 7 *Let P in $\mathbb{R}^2$ be a polygon having a side not parallel to any other side. Then P is L^2-regular.*

Lemma 8 *Let P in $\mathbb{R}^2$ be a convex polygon with a pair of parallel sides having different lengths. Then P is L^2-regular.*

Lemma 9 *Let P in $\mathbb{R}^2$ be a convex polygon which cannot be inscribed in a circle. Then P is L^2-regular.*

4 Notation and preliminary arguments

In the remainder of the paper, a polygon P is given by its vertex set $\{P_h\}_{h=1}^{s}$, where it is assumed that the numbering indicates counterclockwise ordering of the vertices; we write $P \sim \{P_h\}_{h=1}^{s}$. For convenience we use periodic labeling; i.e., P_{h+s}, P_{h+2s}, $\ldots$ refer to the same point P_h for $1 \leq h \leq s$. For every h let

$$\tau_h := \frac{P_{h+1} - P_h}{|P_{h+1} - P_h|}$$

be the direction of the oriented side $P_h P_{h+1}$ and $\ell_h := |P_{h+1} - P_h|$ its length.

For every h let ν_h be the outward unit normal vector corresponding to the side $P_h P_{h+1}$. Let

$$\mathcal{L}_h := |P_h + P_{h+1}|$$

be the length of the vector $P_h + P_{h+1}$. Observe that if $|P_h| = |P_{h+1}|$ (in particular if the polygon P is inscribed in a circle centred at the origin) then

$$P_h + P_{h+1} = \mathcal{L}_h \nu_h \ .$$

We shall always assume $\ell_h \geq 1$ and $\mathcal{L}_h \geq 1$.

Let $\nu(s)$ be the outward unit normal vector at a point $s \in \partial P$ which is not a vertex of P. By applying Green's formula we see that, for any $\rho \geq 1$, we have

$$\begin{aligned}
&\widehat{\chi_P}(\rho\Theta) \\
&= \int_P e^{-2\pi i \rho \Theta \cdot t} \ dt = -\frac{1}{2\pi i \rho} \int_{\partial P} e^{-2\pi i \rho \Theta \cdot s} \ (\Theta \cdot \nu(s)) \ ds \\
&= -\frac{1}{2\pi i \rho} \sum_{h=1}^{s} \ell_h \left(\Theta \cdot \nu_h\right) \int_0^1 e^{-2\pi i \rho \Theta \cdot (P_h + \lambda(P_{h+1} - P_h))} \ d\lambda \\
&= -\frac{1}{4\pi^2 \rho^2} \sum_{h=1}^{s} \frac{\Theta \cdot \nu_h}{\Theta \cdot \tau_h} \ \left[e^{-2\pi i \rho \Theta \cdot P_{h+1}} - e^{-2\pi i \rho \Theta \cdot P_h} \right] \\
&= -\frac{1}{4\pi^2 \rho^2} \sum_{h=1}^{s} \frac{\Theta \cdot \nu_h}{\Theta \cdot \tau_h} \ e^{-\pi i \rho \Theta \cdot (P_{h+1} + P_h)} \left[e^{-\pi i \rho \Theta \cdot (P_{h+1} - P_h)} - e^{\pi i \rho \Theta \cdot (P_{h+1} - P_h)} \right] \\
&= \frac{i}{2\pi^2 \rho^2} \sum_{h=1}^{s} \frac{\Theta \cdot \nu_h}{\Theta \cdot \tau_h} \ e^{-\pi i \rho \mathcal{L}_h \Theta \cdot \nu_h} \sin(\pi \rho \ell_h \Theta \cdot \tau_h) \ . \qquad (14)
\end{aligned}$$

For any $1 \leq h \leq s$, let $\theta_h \in [0, 2\pi)$ be the angle defined by

$$\tau_h =: (\cos\theta_h, \sin\theta_h) \ . \qquad (15)$$

Hence

$$\nu_h = (\sin\theta_h, -\cos\theta_h) \qquad (16)$$

and, if $\Theta := (\cos\theta, \sin\theta)$,

$$\Theta \cdot \tau_h = \cos(\theta - \theta_h) \ , \qquad \Theta \cdot \nu_h = -\sin(\theta - \theta_h).$$

Then (14) can be written as

$$\widehat{\chi_P}(\rho\Theta) = -\frac{i}{2\pi^2 \rho^2} \sum_{h=1}^{s} \frac{\sin(\theta - \theta_h)}{\cos(\theta - \theta_h)} \ e^{\pi i \rho \mathcal{L}_h \sin(\theta - \theta_h)} \sin(\pi \rho \ell_h \cos(\theta - \theta_h))$$

and the equality in (8) yields

$$\begin{aligned}
&\|D_P^\rho\|^2_{L^2(SO(2)\times\mathbb{T}^2)} \qquad (17) \\
&= \rho^4 \sum_{0 \neq k \in \mathbb{Z}^2} \int_0^{2\pi} \left|\widehat{\chi_P}(\rho\,|k|\,\Theta)\right|^2 \ d\theta \\
&= c \sum_{0 \neq k \in \mathbb{Z}^2} \frac{1}{|k|^4}
\end{aligned}$$

$$\times \int_0^{2\pi} \left| \sum_{h=1}^{s} \frac{\sin(\theta-\theta_h)}{\cos(\theta-\theta_h)} \, e^{-\pi i \rho |k| \mathcal{L}_h \sin(\theta-\theta_h)} \sin(\pi\rho\,|k|\,\ell_h \cos(\theta-\theta_h)) \right|^2 d\theta \ .$$

For $P \in \mathfrak{P}$, relation (17) can be further simplified. Let $P \in \mathfrak{P}$ have $s = 2n$ sides (i.e. $P \sim \{P_h\}_{h=1}^{2n}$) and be inscribed in a circle centered at the origin. Then $P_h P_{h+1} = -P_{n+h}P_{n+h+1}$ for any $1 \le h \le n$ and $P_{h+1} + P_h = \mathcal{L}_h \nu_h$. Therefore, for every $1 \le h \le n$,

$$\tau_h = -\tau_{n+h} \ , \qquad \nu_h = -\nu_{n+h} \ , \qquad \ell_h = \ell_{n+h} \ , \qquad \mathcal{L}_h = \mathcal{L}_{n+h} \ .$$

Then the relation (14) becomes

$$\widehat{\chi}_P(\rho\Theta) = \frac{1}{\pi^2\rho^2} \sum_{h=1}^{n} \frac{\sin(\theta-\theta_h)}{\cos(\theta-\theta_h)} \sin(\pi\rho\mathcal{L}_h \sin(\theta-\theta_h)) \sin(\pi\rho\ell_h\cos(\theta-\theta_h)) \ . \tag{18}$$

and the equality in (8) yields

$$\begin{aligned} &\|D_P^\rho\|^2_{L^2(SO(2)\times\mathbb{T}^2)} \\ &= c \sum_{0\neq k\in\mathbb{Z}^2} \frac{1}{|k|^4} \\ &\times \int_0^{2\pi} \left| \sum_{h=1}^{n} \frac{\sin(\theta-\theta_h)}{\cos(\theta-\theta_h)} \sin(\pi\rho|k|\ell_h\cos(\theta-\theta_h)) \sin(\pi\rho|k|\mathcal{L}_h \sin(\theta-\theta_h)) \right|^2 d\theta \\ &\le c \sum_{0\neq k\in\mathbb{Z}^2} \frac{1}{|k|^4} \sum_{h=1}^{n} \int_0^{\pi/2} \left| \frac{\sin(\pi\rho|k|\ell_h\sin\theta)}{\sin\theta} \sin(\pi\rho|k|\mathcal{L}_h\cos\theta) \right|^2 d\theta \ . \end{aligned} \tag{19}$$

The last relation holds for every $P \in \mathfrak{P}$ with $2n$ sides.

5 Proofs

Proof of Lemma 7. The proof of Lemma 7 is essentially the proof of [6, Theorem 3.7], which is stated for a simplex but the argument also works for every polyhedron having a face not parallel to any other face. ■

Proof of Lemma 8. By Lemma 7 we can assume that $P \sim \{P_h\}_{h=1}^{2n}$ is a convex polygon with an even number of sides, and that for every $h = 1, \ldots, n$ the sides P_hP_{h+1} and $P_{h+n}P_{h+n+1}$ are parallel. Suppose that the length ℓ_j of the jth side P_jP_{j+1} is longer than the length ℓ_{j+n} of the opposite side $P_{j+n}P_{j+n+1}$. Then there exist $0 < \varepsilon < 1$ and $0 < \alpha < 1$ such that

$$(1+\varepsilon)\frac{\ell_{j+n}}{\ell_j} < \alpha \ . \tag{20}$$

Let $H > 1$ be a large constant satisfying

$$\sin(\theta-\theta_j) \ge \sqrt{\alpha}\,(\theta-\theta_j) \qquad \text{if } \ 0 \le \theta-\theta_j \le \frac{1+\varepsilon}{H} \ . \tag{21}$$

We further assume (recall $\rho \ge 1$)

$$\frac{1}{H\pi\rho\ell_j} \le \theta - \theta_j \le \frac{1+\varepsilon}{H\pi\rho\ell_j} \ .$$

Observe that (20) and (21) yield

$$
\begin{aligned}
&|\sin(\pi\rho\ell_j \sin(\theta-\theta_j))| - |\sin(\pi\rho\ell_{j+n}\sin(\theta-\theta_{j+n}))| \\
&\geq \sin(\pi\rho\ell_j\sqrt{\alpha}\,(\theta-\theta_j)) - \sin(\pi\rho\ell_{j+n}\,(\theta-\theta_{j+n})) \geq \frac{\alpha}{H} - \frac{1+\varepsilon}{H}\frac{\ell_{j+n}}{\ell_j} =: a_j > 0 \ .
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\left|\frac{\sin(\pi\rho\ell_j\sin(\theta-\theta_j))}{\sin(\theta-\theta_j)}\cos(\theta-\theta_j)e^{-\pi i\rho\Theta\cdot(P_{j+1}+P_j)}\right. \qquad (22)\\
&\left.+\frac{\sin(\pi\rho\ell_{j+n}\sin(\theta-\theta_j))}{\sin(\theta-\theta_j)}\cos(\theta-\theta_{j+n})e^{-\pi i\rho\Theta\cdot(P_{j+n+1}+P_{j+n})}\right| \\
&\geq \frac{|\cos(\theta-\theta_j)|}{|\sin(\theta-\theta_j)|}\left(|\sin(\pi\rho\ell_j\sin(\theta-\theta_j))| - |\sin(\pi\rho\ell_{j+n}\sin(\theta-\theta_{j+n}))|\right) \\
&\geq a_j\ \frac{|\cos(\theta-\theta_j)|}{|\sin(\theta-\theta_j)|}\ .
\end{aligned}
$$

We use the previous estimates to evaluate the last integral in (17) in a neighborhood of θ_j and therefore obtain an estimate from below of $\|D_P^\rho\|_{L^2(SO(2)\times\mathbb{T}^2)}$. By the arguments in [6, Theorem 2.3] or [33, Lemma 10.6], the contribution of all the sides P_hP_{h+1} (with $h\neq j$ and $h\neq j+n$) to the term $\|D_P^\rho\|_{L^2(SO(2)\times\mathbb{T}^2)}$ is $\mathcal{O}(1)$. Then (11), (18) and (22) yield

$$
\|D_P^\rho\|^2_{L^2(SO(2)\times\mathbb{T}^2)} \geq c\int_{\frac{1}{H\pi\rho\ell_j}}^{\frac{1+\varepsilon}{H\pi\rho\ell_j}} \frac{\cos^2\theta}{\sin^2\theta}\,d\theta + c_1 \geq c\int_{\frac{1}{H\pi\rho\ell_j}}^{\frac{1+\varepsilon}{H\pi\rho\ell_j}} \frac{d\theta}{\theta^2} + c_1 \geq c_2\ \rho\ .
$$

■

Proof of Lemma 9. We can assume that $P\sim\{P_h\}_{h=1}^{2n}$ is a convex polygon such that for every $h=1,\ldots,n$ the sides P_hP_{h+1} and $P_{h+n}P_{h+n+1}$ are parallel and of the same length (that is, $\ell_h=\ell_{h+n},\ \tau_h=-\tau_{h+n},\ \nu_h=-\nu_{h+n}$). Then we may assume that P is symmetric about the origin. As P cannot be inscribed in a circle, there exists an index $1\leq j\leq n$ such that the two opposite equal and parallel sides P_jP_{j+1} and $P_{j+n}P_{j+n+1}$ are not the sides of a rectangle. Then P_j+P_{j+1} is not orthogonal to $P_{j+1}-P_j$. Let $\phi_j\in[\theta_j-\pi,\theta_j]$ be defined by

$$
P_{j+1}+P_j=\mathcal{L}_j(\cos\phi_j,\sin\phi_j)\ .
$$

Since $\tau_j=(\cos\theta_j,\sin\theta_j)$ and $\nu(j)=(\cos(\theta_j-\frac{\pi}{2}),\sin(\theta_j-\frac{\pi}{2}))$, see (15) and (16), we have $\phi_j-\theta_j\neq-\frac{\pi}{2}$. We put $\varphi_j:=\phi_j-\theta_j$. Then

$$
\varphi_j\in[-\pi,0]\setminus\{-\frac{\pi}{2}\}\ .
$$

Again we need to find a lower bound for the last integral in (17). As in the previous proof it is enough to consider

$$
\begin{aligned}
F_j(\theta) &:= \sum_{h\in\{j,j+n\}}\frac{\sin(\theta-\theta_h)}{\cos(\theta-\theta_h)}\sin(\pi\rho\ell_j\cos(\theta-\theta_h))e^{-\pi i\rho\Theta\cdot(P_{h+1}+P_h)} \\
&= \frac{\sin(\theta-\theta_j)}{\cos(\theta-\theta_j)}\sin(\pi\rho\ell_j\cos(\theta-\theta_j))\left[e^{-\pi i\rho\mathcal{L}_j\cos(\theta-\phi_j)}-e^{\pi i\rho\mathcal{L}_j\cos(\theta-\phi_j)}\right]
\end{aligned}
$$

$$= -2i\ \frac{\sin(\theta-\theta_j)}{\cos(\theta-\theta_j)} \sin(\pi\rho\ell_j\cos(\theta-\theta_j))\sin(\pi\rho\mathcal{L}_j\cos(\theta-\phi_j))\ .$$

We write

$$\int_0^{2\pi} \left|\frac{\sin(\theta-\theta_j)}{\cos(\theta-\theta_j)} \sin(\pi\rho\ell_j\cos(\theta-\theta_j))\sin(\pi\rho\mathcal{L}_j\cos(\theta-\phi_j))\right|^2\ d\theta$$
$$= \int_0^{2\pi} \left|\frac{\sin(\pi\rho\ell_j\sin\theta)}{\sin\theta}\cos\theta\sin(\pi\rho\mathcal{L}_j\sin(\theta-\varphi_j))\right|^2\ d\theta\ .$$

We shall integrate θ in a neighborhood of 0 (actually $0 \le \theta \le 1$ suffices). As for φ_j we first assume $\varphi_j \in (-\frac{\pi}{2}, 0]$. Then $\cos\varphi_j > 0$ and $\sin\varphi_j \le 0$. Let $0 < \gamma < 1$ satisfy $\cos\varphi_j > \gamma$. In order to prove that $|\sin(\pi\rho\mathcal{L}_j\sin(\theta-\varphi_j)| \ge c$ we consider two cases.
Case 1: $|\sin(\pi\rho\mathcal{L}_j\sin\varphi_j)| > \gamma/2$.
We need to bound $\sin(\theta-\varphi_j) - |\sin\varphi_j|$. Since $\sin\varphi_j \le 0$ one has

$$\frac{\theta}{2}\cos\varphi_j + \left[1-\frac{\theta^2}{2}\right]|\sin\varphi_j| \le \sin\theta\cos\varphi_j - \cos\theta\sin\varphi_j \le \theta\cos\varphi_j + |\sin\varphi_j|\ .$$

Therefore
$$\frac{\theta}{2}\gamma - \frac{\theta^2}{2} \le \sin(\theta-\varphi_j) - |\sin\varphi_j| \le \theta\ . \tag{23}$$
Let $\rho \ge 1$ and assume
$$\frac{\gamma}{8\pi\rho\mathcal{L}_j} \le \theta \le \frac{\gamma}{4\pi\rho\mathcal{L}_j}\ .$$
We recall that $\mathcal{L}_j \ge 1$. Again we have to estimate $\sin(\theta-\varphi_j) - |\sin\varphi_j|$. By (23) we have

$$0 < \frac{\gamma}{16\pi\rho\mathcal{L}_j} - \frac{\gamma^2}{32(\pi\rho\mathcal{L}_j)^2} < \sin(\theta-\varphi_j) - |\sin\varphi_j| \le \frac{\gamma}{4\pi\rho\mathcal{L}_j}\ .$$

Therefore
$$0 < \pi\rho\mathcal{L}_j\sin(\theta-\varphi_j) - \pi\rho\mathcal{L}_j|\sin\varphi_j| \le \frac{\gamma}{4}\ . \tag{24}$$

Hence the assumption of *Case 1* and (24) yield

$$\begin{aligned}
&|\sin(\pi\rho\mathcal{L}_j\sin(\theta-\varphi_j))| \\
&= |\sin(\pi\rho\mathcal{L}_j\,[\sin(\theta-\varphi_j)+\sin\varphi_j] - \pi\rho\mathcal{L}_j\sin\varphi_j)| \\
&= |\sin(\pi\rho\mathcal{L}_j\,[\sin(\theta-\varphi_j)-|\sin\varphi_j|])\cos(\pi\rho\mathcal{L}_j\sin\varphi_j) \\
&- \cos(\pi\rho\mathcal{L}_j\,[\sin(\theta-\varphi_j)-|\sin\varphi_j|])\sin(\pi\rho\mathcal{L}_j\sin\varphi_j)| \\
&\ge |\sin(\pi\rho\mathcal{L}_j\sin\varphi_j)|\,|\cos(\pi\rho\mathcal{L}_j\sin(\theta-\varphi_j) - \pi\rho\mathcal{L}_j|\sin\varphi_j|)| \\
&- |\sin(\pi\rho\mathcal{L}_j\sin(\theta-\varphi_j) - \pi\rho\mathcal{L}_j|\sin\varphi_j|)| \\
&> \frac{\gamma}{2}\left[1-\frac{\gamma^2}{32}\right] - \frac{\gamma}{4} \\
&> \frac{\gamma}{5}\ .
\end{aligned}$$

Case 2: $|\sin(\pi\rho\mathcal{L}_j\sin\varphi_j)| \le \gamma/2$.

Let ρ be large so that $0 \le \theta \le \frac{3}{2\pi\rho\mathcal{L}_j}$ implies $\sin\theta \ge (1-\delta)\theta$, with $\delta < 1/20$. Then for

$$\frac{1}{\pi\rho\mathcal{L}_j} \le \theta \le \frac{3}{2\pi\rho\mathcal{L}_j} \tag{25}$$

we have

$$\theta(1-\delta)\gamma + \left[1 - \frac{\theta^2}{2}\right] |\sin\varphi_j| \le \sin\theta\cos\varphi_j - \cos\theta\sin\varphi_j \le \theta + |\sin\varphi_j|$$

and

$$\theta\gamma(1-\delta) - \frac{\theta^2}{2} \le \sin(\theta - \varphi_j) - |\sin\varphi_j| \le \theta \ . \tag{26}$$

For ρ large enough we have $\frac{9}{8(\pi\rho\mathcal{L}_j)^2} < \frac{\gamma\delta}{\pi\rho\mathcal{L}_j}$. Then (25) and (26) yield

$$\begin{aligned}\frac{\gamma(1-2\delta)}{\pi\rho\mathcal{L}_j} &< \frac{\gamma(1-\delta)}{\pi\rho\mathcal{L}_j} - \frac{9}{8(\pi\rho\mathcal{L}_j)^2} \le \theta\gamma\,(1-\delta) - \frac{\theta^2}{2} \\ &< \sin(\theta-\varphi_j) - |\sin\varphi_j| \le \theta \le \frac{3}{2\pi\rho\mathcal{L}_j}\end{aligned}$$

and

$$\gamma(1-2\delta) < \pi\rho\mathcal{L}_j \sin(\theta-\varphi_j) - \pi\rho\mathcal{L}_j |\sin\varphi_j| \le \frac{3}{2} \ . \tag{27}$$

We choose γ small enough so that

$$\sin(\gamma(1-2\delta)) \ge (1-2\delta)^2\gamma \quad \text{and} \quad \gamma^2/4 < 2\delta \ .$$

Then (27) and the assumption of *Case 2* yield

$$\begin{aligned}&|\sin(\pi\rho\mathcal{L}_j \sin(\theta-\varphi_j))| = |\sin(\pi\rho\mathcal{L}_j\,[\sin(\theta-\varphi_j) - \sin\varphi_j] + \pi\rho\mathcal{L}_j\sin\varphi_j)| \\ &\ge |\cos(\pi\rho\mathcal{L}_j\sin\varphi_j)|\,|\sin(\pi\rho\mathcal{L}_j\sin(\theta-\varphi_j) - \pi\rho\mathcal{L}_j|\sin\varphi_j|)| - |\sin(\pi\rho\mathcal{L}_j\sin\varphi_j)| \\ &\ge \gamma(1-2\delta)^2\sqrt{1-\frac{\gamma^2}{4}} - \frac{\gamma}{2} > \gamma\left[(1-2\delta)^{5/2} - \frac{1}{2}\right] > \frac{\gamma}{4} \ .\end{aligned}$$

Case 1 and *Case 2* prove that for a suitable choice of $0 < \gamma < 1$, such that $\cos\varphi_j > \gamma$, there exist $0 < \alpha < \beta$ such that for $\frac{\alpha}{\pi\rho\mathcal{L}_j} \le \theta \le \frac{\beta}{\pi\rho\mathcal{L}_j}$ and ρ large enough we have

$$|\sin(\pi\rho\mathcal{L}_j\sin(\theta-\varphi_j))| > \frac{\gamma}{5} \ . \tag{28}$$

If $\varphi_j \in [-\pi, -\frac{\pi}{2})$ we have $\cos\varphi_j < 0$ and $\sin\varphi_j \le 0$. Then for $0 \le \theta < 1$ we have

$$-\theta + \left[1 - \frac{\theta^2}{2}\right] |\sin\varphi_j| \le \sin\theta\cos\varphi_j - \cos\theta\sin\varphi_j \le -\sin\theta|\cos\varphi_j| + |\sin\varphi_j| \ .$$

Hence, for a positive constant K,

$$\sin\theta|\cos\varphi_j| \le |\sin\varphi_j| - \sin(\theta-\varphi_j) \le K\ \theta \ .$$

If we choose a suitable constant $\gamma > 0$ such that $|\cos\varphi_j| > \gamma$, we can prove as for the case $\varphi_j \in (-\frac{\pi}{2}, 0]$ that (28) still holds for $\frac{\alpha}{\pi\rho\mathcal{L}_j} \le \theta \le \frac{\beta}{\pi\rho\mathcal{L}_j}$, with $0 < \alpha < \beta$ and ρ large enough. Then (28) yields

$$\int_0^{2\pi} |F_j(\theta)|^2\,d\theta \ge \int_{\frac{\alpha}{\pi\rho\mathcal{L}_j}}^{\frac{\beta}{\pi\rho\mathcal{L}_j}} \left|\frac{\sin(\pi\rho\ell_j\sin\theta)}{\sin\theta}\cos\theta\sin(\pi\rho\mathcal{L}_j\sin(\theta-\varphi_j))\right|^2 d\theta$$

$$\geq c\ \gamma^2 \int_{\frac{\alpha}{\pi\rho\mathcal{L}_j}}^{\frac{\beta}{\pi\rho\mathcal{L}_j}} \left|\frac{\sin(\pi\rho\ell_j \sin\theta)}{\sin\theta}\right|^2 d\theta \geq c_1\ \rho \int_{c_1}^{c_2} \left|\frac{\sin(t)}{t}\right|^2 dt \geq c_2\ \rho.$$

This ends the proof. ■

The proof of Theorem 3 will be complete after the proof of Proposition 4. We need a simultaneous approximation lemma from [29].

Lemma 10 *Let $r_1, r_2, ..., r_n \in \mathbb{R}$. For every positive integer j there exists $j \leq q \leq j^{n+1}$ such that $\|r_s q\| < j^{-1}$ for any $1 \leq s \leq n$, where $\|x\|$ denotes the distance of a real number x from the integers.*

Proof of Proposition 4. Let $P \sim \{P_j\}_{j=1}^{2n}$ be a polygon in $\mathfrak{P}$. For every positive integer u let

$$A_u^j := \{k \in \mathbb{Z}^2 : 0 < \mathcal{L}_j|k| \leq u^2\} \quad \text{for } j = 1, \ldots, n\ , \qquad A_u := \bigcup_{j=1}^{n} A_u^j\ .$$

Observe that $\mathrm{card}(A_u^j) \leq 4u^4$ and therefore $\mathrm{card}(A_u) \leq 4nu^4$. By Lemma 10 there exists a sequence $\{\rho_u\}_{u=1}^{+\infty}$ of positive integers such that, for every $k \in A_u$ and every $j = 1, \ldots, n$,

$$u \leq \rho_u \leq u^{4nu^4+1}\ , \qquad |\sin(\pi\rho_u|k|\mathcal{L}_j)| < 1/u\ . \tag{29}$$

Observe that (29) implies

$$u \geq c_\varepsilon \log^{\frac{1}{4+\varepsilon}}(\rho_u) \tag{30}$$

for every $\varepsilon > 0$. For any $1 \leq j \leq n$ and $k \in A_u^j$ we split the integral in (19) into several parts.

$$E_{1,j,|k|}^{\rho} := \int_0^{(8\rho_u|k|)^{-1}} \left|\frac{\sin(\pi\rho_u|k|\ell_j \sin\theta)}{\sin\theta} \sin(\pi\rho_u|k|\mathcal{L}_j \cos\theta)\right|^2 d\theta\ .$$

For $0 \leq \theta \leq (8\rho_u|k|)^{-1}$ we have $0 \leq 1 - \cos\theta \leq \left(128\rho_u^2|k|^2\right)^{-1}$. Then (29) yield

$$\begin{aligned}
&|\sin(\pi\rho_u|k|\mathcal{L}_j \cos\theta)| \\
&= |\sin(\pi\rho_u|k|\mathcal{L}_j\,[\cos\theta - 1 + 1])| \\
&\leq |\sin(\pi\rho_u|k|\mathcal{L}_j(\cos\theta - 1))\cos(\pi\rho_u|k|\mathcal{L}_j)| \\
&+ |\sin(\pi\rho_u|k|\mathcal{L}_j)\cos(\pi\rho_u|k|\mathcal{L}_j(\cos\theta - 1)| \\
&\leq |\sin(\pi\rho_u|k|\mathcal{L}_j(1 - \cos\theta))| + |\sin(\pi\rho_u|k|\mathcal{L}_j)| \\
&\leq \frac{\pi\mathcal{L}_j}{128\rho_u|k|} + |\sin(\pi\rho_u|k|\mathcal{L}_j)| \\
&\leq c\ \frac{1}{u}\ .
\end{aligned} \tag{31}$$

By (31) we obtain

$$\begin{aligned}
E_{1,j,|k|}^{\rho} &\leq c\ \frac{1}{u^2} \int_0^{(8\rho_u|k|)^{-1}} \left|\frac{\sin(\pi\rho_u|k|\ell_j \sin\theta)}{\sin\theta}\right|^2 d\theta \\
&\leq c_1\ \frac{|k|\rho_u}{u^2} \int_0^1 \left|\frac{\sin(t)}{t}\right|^2 dt \leq c_2\ \frac{|k|\rho_u}{u^2}\ .
\end{aligned}$$

Let

$$E^{\rho}_{2,j,|k|} := \int_{(8\rho_u|k|)^{-1}}^{\left(8u^{1/4}\rho_u^{1/2}|k|^{1/2}\right)^{-1}} \left| \frac{\sin(\pi\rho_u|k|\ell_j \sin\theta)}{\sin\theta} \sin(\pi\rho_u|k|\mathcal{L}_j \cos\theta) \right|^2 d\theta \ .$$

For $(8\rho_u|k|)^{-1} \leq \theta \leq \left(8\ell^{1/4}\rho_u^{1/2}\,|k|^{1/2}\right)^{-1}$ we have

$$\frac{1}{2000\rho_u^2|k|^2} \leq 2\sin^2(\theta/2) = 1-\cos\theta \leq \frac{1}{128u^{1/2}\rho_u|k|} \ .$$

As in (31) we obtain

$$\begin{aligned} |\sin(\pi\rho_u|k|\mathcal{L}_j\cos\theta)| &\leq |\sin(\pi\rho_u|k|\mathcal{L}_j(1-\cos\theta))| + |\sin(\pi\rho_u|k|\mathcal{L}_j)| \\ &\leq \frac{\pi\mathcal{L}_j}{128u^{1/2}} + \frac{1}{u} \leq c\ u^{-1/2} \end{aligned}$$

and then

$$\begin{aligned} E^{\rho}_{2,j,|k|} &\leq c\ \frac{1}{u} \int_{(8\rho_u|k|)^{-1}}^{\left(8u^{1/4}\rho_u^{1/2}|k|^{1/2}\right)^{-1}} \left| \frac{\sin(\pi\rho_u|k|\ell_j\sin\theta)}{\sin\theta} \right|^2 d\theta \\ &\leq c_1\ \frac{1}{u} \int_{(8\rho_u|k|)^{-1}}^{\left(8u^{1/4}\rho_u^{1/2}|k|^{1/2}\right)^{-1}} \frac{d\theta}{\theta^2} \leq c_2\ \frac{\rho_u|k|}{u} \ . \end{aligned}$$

Let $1/4 < \lambda < 1/2$ and let

$$E^{\rho}_{3,j,|k|} := \int_{\left(8u^{1/4}\rho_u^{1/2}|k|^{1/2}\right)^{-1}}^{\lambda} \left| \frac{\sin(\pi\rho_u|k|\ell_j\sin\theta)}{\sin\theta} \sin(\pi\rho_u|k|\mathcal{L}_j\cos\theta) \right|^2 d\theta \ .$$

We have

$$E^{\rho}_{3,j,|k|} \leq \int_{\left(8u^{1/4}\rho_u^{1/2}|k|^{1/2}\right)^{-1}}^{\lambda} \frac{d\theta}{\theta^2} \leq 8u^{1/4}\rho_u^{1/2}\,|k|^{1/2} \ .$$

Finally we have

$$E^{\rho}_{4,j,|k|} := \int_{\lambda}^{\frac{\pi}{2}} \left| \frac{\sin(\pi\rho_u|k|\ell_j\sin\theta)}{\sin\theta} \sin(\pi\rho_u|k|\mathcal{L}_j\cos\theta) \right|^2 d\theta \leq c \ .$$

By the above estimates, (19), (29) and (30) we have

$$\begin{aligned} &\|D_P^{\rho_u}\|^2_{L^2(SO(2)\times\mathbb{T}^2)} \\ &\leq c\ \rho_u \sum_{k\in A_u} \frac{1}{|k|^3} \left(\frac{1}{u^2} + \frac{1}{u} + u^{1/4}\rho_u^{-1/2}\,|k|^{-1/2} + \rho_u^{-1}\,|k|^{-1} \right) \\ &+ c_1 \sum_{k\notin A_u} \frac{1}{|k|^4} \int_0^{\pi/2} \left| \frac{\sin(\pi\rho_u|k|\ell_j\sin\theta)}{\sin\theta} \right|^2 \ d\theta \\ &\leq c\ \rho_u \sum_{0\neq k\in A_u} \frac{1}{|k|^3} u^{-1/4} \\ &+ c_1 \sum_{|k|>c_1u^2} \frac{1}{|k|^4} \left(\int_0^{(\rho_u|k|)^{-1/2}} (\rho_u\,|k|)\ d\theta + \int_{(\rho_u|k|)^{-1/2}}^{\pi/2} \frac{1}{\theta^2}\ d\theta \right) \end{aligned}$$

$$\leq c_\varepsilon \ \rho_u \sum_{0\neq k\in\mathbb{Z}^2} \frac{1}{|k|^3} \log^{-\frac{1}{16+\varepsilon}}(\rho_u) + c \sum_{|k|>c_1 u^2} \frac{1}{|k|^4} (\rho_u\,|k|)^{1/2}$$
$$\leq c_\varepsilon \ \rho_u \log^{-\frac{1}{16+\varepsilon}}(\rho_u) + c\ \rho_u^{1/2} \int_{\{t\in\mathbb{R}^2:|t|>c_1u^2\}} \frac{1}{|t|^{7/2}}\ dt$$
$$\leq c_\varepsilon \ \rho_u \log^{-\frac{1}{16+\varepsilon}}(\rho_u) \ .$$

■

We now turn to the proof of Proposition 6, which depends on the following lemma proved by L. Parnovski and A. Sobolev [29].

Lemma 11 *For any $\varepsilon > 0$ there exist $\rho_0 \geq 1$ and $0 < \alpha < 1/2$ such that for every $\rho \geq \rho_0$ there exists $k \in \mathbb{Z}^d$ such that $|k| \leq \rho^\varepsilon$ and $\|\rho|k|\| \geq \alpha$, where $\|x\|$ is the distance of a real number x from the integers.*

Proof of Proposition 6. Let $P \sim \{P_j\}_{j=1}^{2n}$ be a polygon in $\mathfrak{P}$. Let $\varepsilon > 0$ and let $j \in \{1, 2, \dots, n\}$. By Lemma 11 there exist $\rho_0 \geq 1$ and $0 < a < 1/2$ such that for any $\rho \geq \rho_0$ there is $\widetilde{k} \in \mathbb{Z}^2$ such that $|\widetilde{k}| \leq \rho^{\varepsilon/3}$ and $|\sin(\pi\rho|\widetilde{k}|\mathcal{L}_j)| > a$. Then we consider the interval

$$\theta_j \leq \theta \leq \theta_j + \frac{1}{\pi\rho|\widetilde{k}|} \ . \tag{32}$$

We have

$$0 \leq 1 - \cos(\theta - \theta_j) \leq \frac{1}{2(\pi\rho|\widetilde{k}|)^2} \ .$$

Then for large ρ we have

$$\begin{aligned} & |\sin(\pi\rho|\widetilde{k}|\mathcal{L}_j \cos(\theta-\theta_j))| \\ & \geq |\sin(\pi\rho|\widetilde{k}|\mathcal{L}_j)||\cos(\pi\rho|\widetilde{k}|\mathcal{L}_j(1-\cos(\theta-\theta_j)))| \\ & - |\sin(\pi\rho|\widetilde{k}|\mathcal{L}_j(1-\cos(\theta-\theta_j)))| \\ & \geq c \left[1 - \frac{\mathcal{L}_j^2}{8(\pi\rho|\widetilde{k}|)^2}\right] - \frac{\mathcal{L}_j}{2\pi\rho|\widetilde{k}|} > c_1 \ . \end{aligned} \tag{33}$$

As before the sides non parallel to P_jP_{j+1} give a bounded contribution to the integration of $\left|D^\rho_{P_n}\right|^2$ over the interval in (32). Finally (33) yields

$$\begin{aligned} & \|D^\rho_{P_n}\|^2_{L^2(SO(2)\times\mathbb{T}^2)} \\ & \geq c + c_1 \frac{1}{|\widetilde{k}|^4} \int_{\theta_j}^{\theta_j+1/(\pi\rho|\widetilde{k}|)} \left|\frac{\sin(\pi\rho|\widetilde{k}|\ell_j\sin(\theta-\theta_j))}{\sin(\theta-\theta_j)}\right|^2 \\ & \times |\sin(\pi\rho|\widetilde{k}|\mathcal{L}_j\cos(\theta-\theta_j))|^2|\cos(\theta-\theta_j)|^2\ d\theta \\ & \geq c + c_2 \frac{1}{|\widetilde{k}|^4} \int_0^{1/(\pi\rho|\widetilde{k}|)} \left|\frac{\sin(\pi\rho|\widetilde{k}|\ell_j\sin\theta)}{\sin\theta}\right|^2\ d\theta \\ & \geq c + c_3 \frac{\rho}{|\widetilde{k}|^3} \\ & \geq c_4\ \rho^{1-\varepsilon} \ . \end{aligned}$$

■

The proofs of Lemmas 7, 8, and 9 actually show that $\|\widehat{\chi_P}(\rho\cdot)\|_{L^2(\Sigma_1)} \geq c\,\rho^{-3/2}$ whenever $P \notin \mathfrak{P}$. Hence Theorem 3 and Proposition 1 readily yield the following result.

Corollary 12 *Let P be a polygon in $\mathbb{R}^2$. Then P satisfies*

$$\|\widehat{\chi_P}(\rho\cdot)\|_{L^2(\Sigma_1)} \geq c\,\rho^{-3/2}$$

if and only if $P \notin \mathfrak{P}$.

The results in this paper (apart from Lemma 7) seem to be tailored for the planar case. A different (perhaps simpler) approach might be necessary in order to deal with the multi-dimensional cases.

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Giancarlo Travaglini
Dipartimento di Statistica e Metodi Quantitativi
Edificio U7, Università di Milano-Bicocca
Via Bicocca degli Arcimboldi 8, 20126, Milano, Italia
giancarlo.travaglini@unimib.it

Maria Rosaria Tupputi,
Dipartimento di Statistica e Metodi Quantitativi
Edificio U7, Università di Milano-Bicocca,
Via Bicocca degli Arcimboldi 8, 20126 Milano, Italia
maria.tupputi@unimib.it